COUNTING PROBLEMS

This packet was created by Vaag Mosca, a mathematics teacher and coach of Math Team and MathCounts at the William Diamond Middle School in Lexington, Massachusetts in order to assist students and coaches who want to increase their knowledge about problems involving counting, as well as to improve their strategies, speed and accuracy in solving them.

The topic of "Counting Problems" was new to the math team experience this year. As an experiment, the ARITHMETIC category for the fifth competition of the year (April 1) was focused on problems similar to those in this packet. In addition, for the Arithmetic category for April, there were problems involving probability, but **not** problems involving statistics.

The instruction, examples, and problems contained in the following document apply to two forums – MathCounts and the Intermediate Math League of Eastern Massachusetts (IMLEM). For the IMLEM, this packet provides a detailed description of the subtopics of Counting Problems. For MathCounts, much credit is given to the work which was published in the 2002-2003 MathCounts Handbook. The "Representation of Combinatorics" page is included in this packet. The solutions to this page, as well as the solutions to all of the problems in the MathCounts handbooks of the past two years, were written by our own Kent Findell, who is the contest author for our IMLEM.

9/17/03

This has been edited for web presentation in April 2006 by the IMLEM webmaster, who takes responsibility for any errors or omissions from the original. As of 2005-06, Mr. Mosca is Director of Mathematics for the Belmont Public Schools.

Errors or suggestions? Please contact the webmaster at www.imlem.org

The categories of “counting” problems in this unit include the following six sub-categories:

1) The Counting Principle
2) Powers
3) Factorials
4) Permutations
5) Combinations
6) Pascal’s Triangle

1) The Counting Principle

Property: If an event can occur in M number of ways, and another event can occur in N number of ways, then the number of ways the two events can occur either simultaneously (at the same time) or in succession (one after the other) is the product MN.

Example: A cubical die has six surfaces, numbered 1,2,3,4,5,6.
A coin has two sides, namely heads and tails. If a die is rolled and a coin is flipped, then there are (6)(2), or 12 possible outcomes.

Here are three ways to illustrate that there are, in fact, 12 outcomes – a list of the 12 outcomes as i) ordered pairs, ii) a rectangular array, and iii) a tree diagram.

i) Ordered pairs

Let H = heads and T = tails. The list of 12 outcomes is { (1,H), (2,H), (3,H), (4,H),(5,H),(6,H),(1,T),(2,T),(3,T),(4,T),(5,T),(6,T) }

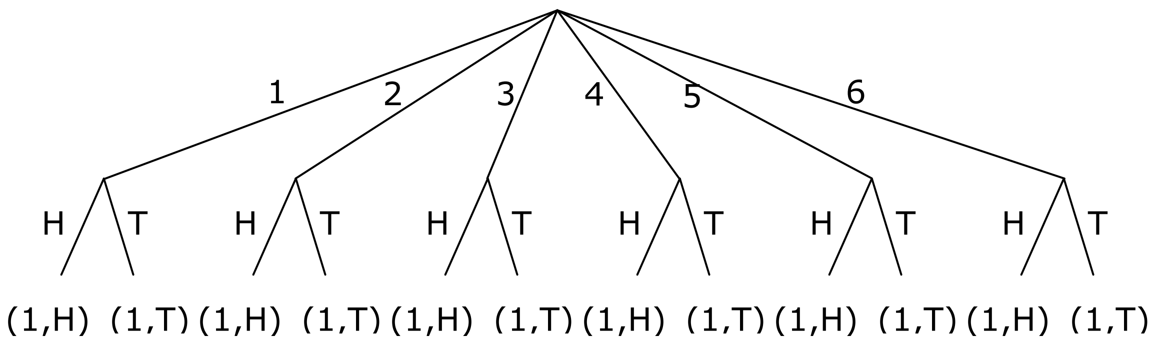
There are 6x2, or 12 ordered pairs in the outcome set.

ii) Rectangular array

	1	2	3	4	5	6
T	(1,T)	(2,T)	(3,T)	(4,T)	(5,T)	(6,T)
H	(1,H)	(2,H)	(3,H)	(4,H)	(5,H)	(6,H)

iii) Tree Diagram

Each of the six branches at the top of the diagram represents the six outcomes of a die roll. Each of these six branches then branches out to the two outcomes of a coin flip. In all, there are 6x2, or 12 ordered pairs in the outcome set.



Conclusion – These three visual displays support the simple idea that if the number of outcomes of individual events is known, then the product of those numbers represents the number of outcomes of those events as they occur either simultaneously or in succession. In other words,

if A = the number of outcomes for event A,
and B = the number of outcomes for event B,
and C = the number of outcomes for event C, etc.,

then the product (A)(B)(C) … represents the total number of outcomes for those events as if they had either occurred at the same time (simultaneously) or one after the other (in succession).

2) Powers

Definition: $\mathbf{X^y = \underbrace{X\cdot X\cdot X\cdot \ldots \cdot X}_{\text{Y factors of X}}}$

Example: $3^4 = 3x3x3x3 = 81$

Problem 1: How many 3-letter monograms (for example, initials on a towel or a briefcase) may be written, if each of the three letters is chosen from the 26 letters of the alphabet?

Solution 1: Since it is not mentioned whether or not letters may be repeated, we assume that they can be repeated.

$26^3 = 26x26x26 = 17576$

Problem 2: How many 6-letter license plates are possible, if each character must be a letter, but the first letter cannot be the letter O?

Solution 2: 25x265 = 25x26x26x26x26x26 = 297,034,400

Problem 3: How many 5-digit numbers are possible, if they are all even numbers, and the first digit is not 0?

Solution 3: $9 \times 10^3 \times 5 = 9 \times 10 \times 10 \times 10 \times 5 = 45,000$

3) Factorials

Definition: **$N! = N(N-1)(N-2)(N-3) \ldots (3)(2)(1)$**

Example: Evaluate 5!

Solution: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Problem 1: How many ways can four students be seated in a row which has four desks?

Solution 1: $4! = 4 \times 3 \times 2 \times 1$

Problem 2: How many ways can seven books be placed on a shelf which has space for all seven books?

Solution 2: $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Problem 3: Using only the digits 1,2,5,7,8 and 9, how many 6-digit numbers can be written if no digit is repeated?

Solution 3: $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

4) Permutations

Definition: The word *permutation* means "arrangement."

Definition: $_{n}P_{r}$ represents the number of ways in which n number of objects can be arranged into r number of places.

Formula: $$_{n}P_{r} = \frac{n!}{(n-r)!}$$

Problem 1: Evaluate $_{5}P_{2}$

Solution 1: $$_{5}P_{2} = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$$

$$= \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

$$= 5 \cdot 4 = 20$$

$_{5}P_{2}$ answers the question, "How many ways can 5 objects, like people, be arranged into 2 places, like seats at a table?"

** A simpler calculation acknowledges that any of the five people can occupy the first seat, and any of the remaining four people can occupy the other seat.
5x4 = 20

Problem 2: Evaluate ${}_6P_4$

Solution 2:

$$
\begin{aligned}
{}_6P_4 &= \frac{6!}{(6-4)!} = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1} \\
&= 6\cdot 5\cdot 4\cdot 3 \\
&= 360
\end{aligned}
$$

${}_6P_4$ answers the question, “How many ways can 6 objects, like books, be arranged on a shelf which only has space for four books?”

** A Simpler calculation acknowledges that any of the eix books can occupy the first position on the shelf, leaving five books which could occupy the second position, leaving four which could occupy the third position, and leaving three which could occupy the fourth position. 6x5x4x3 = 360.

Problem 3: Evaluate ${}_5P_5$

Solution 3:

$$
\begin{aligned}
{}_6P_4 &= \frac{5!}{(5-5)!} \\
&= \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1} \qquad \text{Note: } 0! = 1 \\
&= 120
\end{aligned}
$$

${}_5P_5$ is equivalent to 5!, and ${}_nP_n$ is the same as **n!**

5) Combinations – the “choose” function

Formula:

$$ {}_n\mathbf{C}_r = \binom{n}{r} $$

$$ = \frac{{}_nP_r}{r!} $$

$$ = \frac{n!}{(n-r)!\,r!} $$

Example:

$$ {}_5\mathbf{C}_3 = \frac{{}_5P_3}{3\,!} $$

$$ = \frac{5!}{(5-3)!\,3!} $$

$$ = \frac{60}{6} $$

$$ = 10 $$

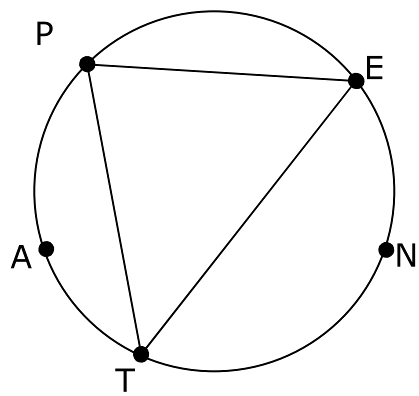
${}_5\mathbf{C}_3$ is sometimes read, “5 choose 3,” and answers the question, “How many ways can three elements be chosen from a set which has five distinct elements?”

Here are some variations of this question:

A) How many 3-element subsets are there for the set {A,B,C,D,E} ?

Solution: $_5C_3 = 10$

B) How many triangles can be drawn in the figure below, where three of the lettered points are vertices of the triangle? One such triangle is already drawn.



Solution: $_5C_3 = 10$

C) How many different ways can a 3-member committee be formed from the members of a 5-member club?

Solution: $_5C_3 = 10$

The information in this section on Combinations was inspired by a page from the MathCounts 2002-2003 handbook called "Representation with Combinatorics."

6) <u>Pascal's Triangle</u>

The numbers in Pascal's Triangle (PT) are as follows:

```
                    1
                 1     1
              1     2     1
           1     3     3     1
        1     4     6     4     1
     1     5    10    10     5     1
  1     6    15    20    15     6     1
```

And so on.

The "outer shell" consists of all ones. Each number inside that shell is the sum of the numbers above / diagonal to it.

Each number in <u>PT</u> is also a <u>combination</u> number, as described in section #5.

So, here are three equivalent forms of the first five rows of Pascal's Triangle:

1) Written just with numbers
2) Written in combination format, and
3) Written in the alternative combination format

<u>Just numbers</u>:

```
              1
           1     1
        1     2     1
     1     3     3     1
  1     4     6     4     1
```

Combination format:

$$
\begin{array}{ccccccccc}
 & & & & {}_0C_0 & & & & \\
 & & & {}_1C_0 & & {}_1C_1 & & & \\
 & & {}_2C_0 & & {}_2C_1 & & {}_2C_2 & & \\
 & {}_3C_0 & & {}_3C_1 & & {}_3C_2 & & {}_3C_3 & \\
{}_4C_0 & & {}_4C_1 & & {}_4C_2 & & {}_4C_3 & & {}_4C_4
\end{array}
$$

Alternative format:

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

From the pattern, the xth entry in the yth row is ${}_{y-1}C_{x-1}$.

Writing numbers in Pascal's Triangle may be a more efficient way to solve some "counting" questions.

For example: How many ways can a two-person committee be chosen from a group of five people?

Solution: Use ${}_5C_2$, which can be calculated either as in section 5 on Combinations, or by writing out the first six rows of Pascal's Triangle, and locating the third entry in the sixth row, which is the number 10:

Row 1						1						
Row 2					1		1					
Row 3				1		2		1				
Row 4			1		3		3		1			
Row 5		1		4		6		4		1		
Row 6	1		5		(10)		10		5		1	

This same problem could have been solved less quickly as in section 5 as follows:

$$
\begin{aligned}
{}_5C_2 &= \frac{{}_5P_2}{2\,!} \\
&= \frac{5!}{(5-2)!\,2!} \\
&= \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 2\cdot 1} = \frac{120}{12} \\
&= 10
\end{aligned}
$$

Pascal's Triangle can also be used to answer questions involving events which have two outcomes which are equally likely, such as flipping a coin (heads ot tails) or having a child (girl or boy) or having to choose one of two paths at a fork in the road (left or right.)

The sum of the numbers in each row is a power of two. The first row is 1, which is 2^0, the sum of the numbers in the second row is 2, which is 2^1, the sum of the numbers in the third row is 4, which is 2^2, and so on.

Pascal's Triangle	Sum of the numbers in the row:		Power Of 2:
1	1	=	2^0
1 1	2	=	2^1
1 2 1	4	=	2^2
1 3 3 1	8	=	2^3
1 4 6 4 1	16	=	2^4
1 5 10 10 5 1	32	=	2^5
1 6 15 20 15 6 1	64	=	2^6

The numbers in row two, for example, are 1 1. The first 1 represents the one way in which a girl could be born into a one-child family, while the second 1 represents the one way in which a boy could be born into a one-child family.

The numbers in row three are 1 2 1. The first 1 represents the one way in which two girls could be born into a two-child family (G,G), the 2 in the middle represents the two ways in which a boy and a girl can be born into a two-child family (either (G,B) or (B,G)), and the final 1 represents the one way in which two boys could be born into a two-child family (B,B).

The numbers in row four are 1 3 3 1. The first 1 represents the one way in which three girls could be born into a three-child family (G,G,G), the 3 represents the three ways in which two girls and a boy can be born into a three-child family (either (G,G,B) or (G,B,G) or (B,G,G)), the next 3 represents the three ways in which one girl and two boys can be born into a three-child family (either (G,B,B) or (B,G,B) or (B,B,G)), and the final 1 represents the one way in which three boys could be born into a three-child family (B,B,B).

Once it is understood what the numbers in Pascal's Triangle represent, then the probabilities of some events can be easily calculated. Since the sum of the numbers in each row is a power of two, it is just a matter of pairing the correct number in the row with the power of two of that row, and making a fraction!

For example, if we wanted to know the probability that the five children born into a family is 3 girls and 2 boys, we could represent this question as, "How many ways can three girls be born into a five-child family?"

First, how many ways can there be 3 girls and 2 boys? That is $_5C_3$, which is calculated the long way as a combination, as follows:

$$\begin{aligned} {}_5\mathbf{C}_3 &= \frac{{}_5P_3}{3\,!} \\ &= \frac{5!}{(5-3)!\,3!} \\ &= \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1\cdot 3\cdot 2\cdot 1} = \frac{120}{12} \\ &= 10 \end{aligned}$$

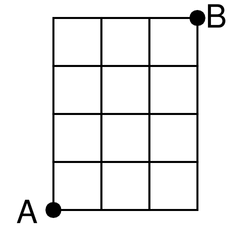
Second, we need the total number of ways there could be 5 children in a family. Since each child's position could have been a boy or a girl, there are $2^5 = 32$ ways. This is also the sum of the entries in the 6^{th} row of Pascal's Triangle, and the power of two that corresponds to this row is $2^5 = 32$.

So the probability is (number of ways we can have 3 girls, 2 boys) $\div$ (number of ways to have 5 children) = 10/32 or 5/16.

Representation with Combinatorics

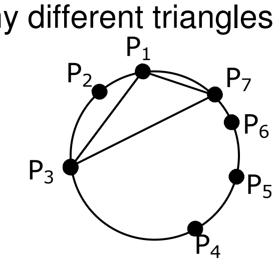
1. _______ How many three-element subsets does the set {A, B, C, D, E, F, G} have?

2. _______ How many different ways can a three-person committee be formed from the members of a seven-member club?

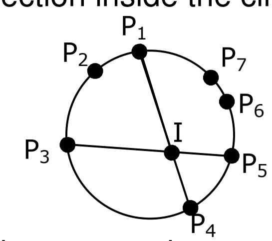


3. _______ Each small square has sides of length 1 unit. How many paths of length seven units are there from A to B in the grid to the right?

4. _______ Let P_1, P_2, P_3, P_4, P_5, P_6 and P_7 be seven different points on a circle. How many different triangles can be formed using these given points as vertices?



5. _______ Seven points are distributed around a circle. All possible chords, using the given seven points as endpoints, are drawn. What is the greatest possible number of points of intersection inside the circle? The point I is an example of one such point.



6. _______ How many solutions are there to $a + b + c + d = 4$, where *a, b, c* and *d* are non-negative integers? For example (2, 1, 0, 1) is such a solution.

7. _______ How many solutions are there to $a + b + c + d = 8$, subject to the condition that each of the variables is a positive integer?

8. _______ What is the coefficient of X^3 in the expanded form of $(X+1)^7$?

9. _______ What is the value of $\frac{7!}{(7-3)!\,3!}$?

10. _______ What is the fourth entry of the eighth row of Pascal's trangle shown below?

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Representation with Combinatorics
Solutions

As you have probably already discovered, the answer to each of the problems in “Representation with Combinatorics” is 35. What you may not have realized is that each of the problems is a different way to represent “seven choose three” or ${}_7C_3$. Each of the solutions below provides an explanation of how ${}_7C_3$ leads to the desired answer.

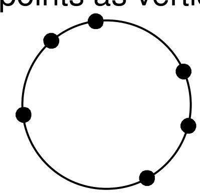
Full solutions may be included here at a later time.

Counting / Combinatorics Stretch MathCounts 2002-03 www.mathcounts.org

1. _______ Bob has stencils to paint the digits 2, 5, and 8. How many distinct three-digit house numbers can he paint, using only the stencils?

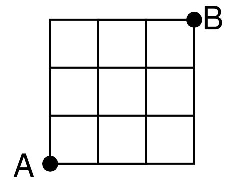
2. _______ At the end of a game, each of the five members of a basketball team shake hands with each of the five members of the other team, and all of the players shake hands with the two referees. How many handshakes occur?

3. _______ Six points are drawn on a circle. How many distinct convex *pentagons* can be drawn using only these points as vertices?



4. ________ A nursery employee wishes to plant six Golden Delicious apple trees and two Bartlett pear trees in one row. How many distinct arrangements are possible?

5. _______ Each small square has sides of length 1 unit. How many distinct paths of length six units are there from A to B?



6. _______ A state with ten million cars plans to issue license plates which consist of any four letters followed by an *n*-digit number. If the state wants to have enough distinct license plates for all of the cars, what is the minimum possible value for *n*?

7. _______ There are six tags numbered 1, 2, 2, 3, 3 and 4. Using these tags, how many distinct three-digit numbers can be formed such that two of the digits are the same?

8. _______ A teacher has made ten statements for a True-False test. Four statements are true and six are false. How many distinct answer keys could there be for the test?

9. _______ How many perfect square factors does the number 46,656 have?

10. ______ There are eight boys and six girls who are members of the trumpet sdection in the band. From the members of the trumpet section, a quintet is being formed. If the quintet must consist of three boys and two girls, how many quintets are possible?

Counting/Combinatorics Stretch – Answers

1. 27 numbers
2. 45 handshakes
3. 6 pentagons (one point is left out for each one)
4. 28 arrangements (eight choose two)
5. 20 paths (three Ups and three Rights – arrange any way. 6 choose 3.)
6. 2 digits
7. 18 possible 3-digit numbers
8. 210 possible answer keys
9. 16 different square factors of 46,656 = $4^3 \times 9^3$, so 4^0x9^0, 4^1x9^0, 4^2x9^0,… 4^0x9^1,4^1x9^1, etc.
10. 840 possible quintets

Full solutions may be included here at a later time.