Intermediate Mathematics League of Eastern Massachusetts

www.imlem.org

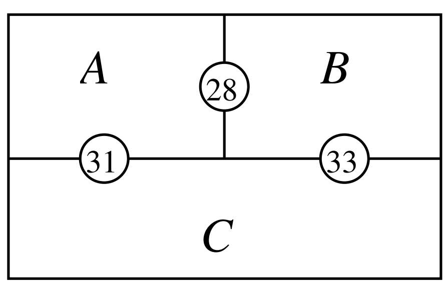
Meet #2
November, 2002

Category 1
Mystery
Meet #2, November, 2002

1. I am a two-digit prime number. The product of my digits is 15. What number am I?

2. Misha bought a candy bar and payed for it with a $50 bill. For change, the cashier gave her three each of three different kinds of bills and three each of four different kinds of coins. What is the price of the candy bar if it is less than five dollars? Express your answer in dollars or cents with the appropriate symbol.

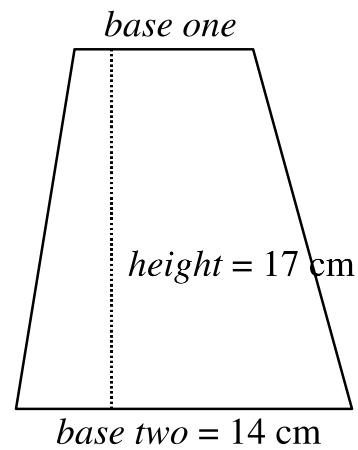
3. Regions *A, B,* and *C* in the figure below each contain a different number. The number in each circle is the sum of the numbers in the two adjacent regions. What is the value of the region with the greatest value?



Answers
1. ________________
2. ________________
3. ________________

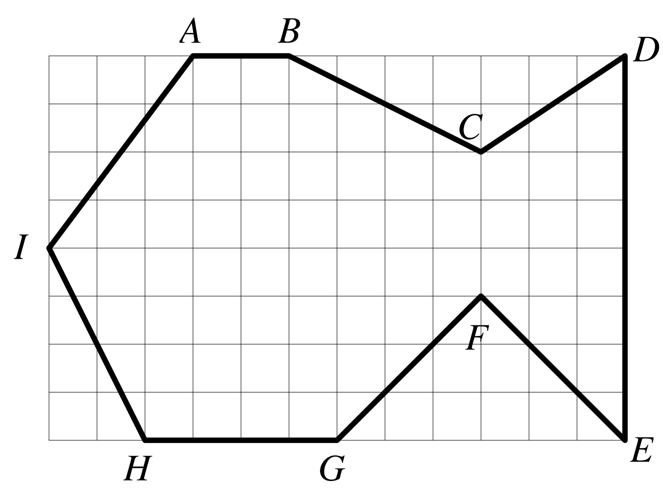
Category 2
Geometry
Meet #2, November, 2002

1. If the trapezoid shown here has an area of 195.5 square centimeters, how many centimeters are in the length of base one?



2. Arnold bought a rectangular sheet of plywood measuring 4 feet by 8 feet. From each of the four corners of the sheet of plywood, he cut rectangles for some shelves. Two of the rectangles measured 12 inches by 36 inches and the other two measured 10 inches by 30 inches. If we disregard any waste from the thickness of the cuts, how many feet are in the perimeter of the remaining piece of plywood?

3. If the side length of each square in the grid below measures 1 centimeter, how many square centimeters are there in the area of polygon *ABCDEFGHI*?



Answers
1. _______________
2. _______________
3. _______________

Category 3
Number Theory
Meet #2, November, 2002

1. Let F be the greatest common factor of 36 and 48.
Let M be the least common multiple of 36 and 48.
What is the product of F and M?

2. If n is a proper factor of 135 and m is a proper factor of 175, what is the largest possible value of $n + m$? (A proper factor of a number is any factor other than the number itself.)

3. John has a rectangular piece of poster board that measures 36 inches by 42 inches. He wants to cut this entire board into some number of squares of equal size with no waste. How many inches are in the side length of the largest possible squares that John can cut from his poster board?

Answers
1. _______________
2. _______________
3. _______________

Category 4
Arithmetic
Meet #2, November, 2002

1. What is $\frac{2}{3}$ of 40% of $\frac{5}{6}$ of 0.7 of 90?

2. Simplify $\dfrac{0.\overline{78}}{0.\overline{91}}$

3. What fraction, in lowest terms, is 65% greater than $\frac{20}{39}$?

Answers
1. ________________
2. ________________
3. ________________

Category 5
Algebra
Meet #2, November, 2002

1. A plumber charged a flat fee of $50 to come to the house, plus an hourly rate for the time he spends working at the house. If the plumber stayed for three and a half hours and charged a total of $274, what is his hourly rate? Express your answer in dollars.

2. Five less than three times the sum of a number and eleven is equal to seventy-three minus four times the sum of the number and six. What is the number?

3. The formula for the surface area of a torus (a donut) is $S = 2\pi r \cdot 2\pi R = 4\pi^2 r R$, where r is the radius of a cross section of the ring and R is the radius from the center of the hole to the center of the ring. (See diagram below.) Imagine a perfect chocolate frosted donut with $r = 0.5$ inches and $R = 1.5$ inches. If exactly half of this donut is covered with chocolate frosting, how many square inches of the surface of the donut is covered with frosting? Use 3.14 for π and round your answer to the nearest tenth.

Answers
1. _______________
2. _______________
3. _______________

Category 6
Team Questions
Meet #2, November, 2002

1. If $2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e = 792$, find the value of $(a+1)(b+1)(c+1)(d+1)(e+1)$.

2. The GCF of two numbers is 18 and the LCM is 324. What is the sum of the two numbers if it is not 342?

3. If $\frac{1}{a}+\frac{1}{b}=\frac{7}{36}$ and $a+b=21$, what is the product of *a* and *b*?

4. What digit is in the 45th decimal place in the decimal equivalent of $\frac{9}{14}$?

5. In the game of bootfall, one player tosses the left boot down the stairs and the other player tosses the right boot. If the boot lands upright, the player earns 7 points. If the boot lands on its side, the player earns 3 points. The winner is whoever has the highest score when the two players get tired of the game. What is the largest score that a player can never obtain in the game of bootfall?

Answers
1. ____________ = *A*
2. ____________ = *B*
3. ____________ = *C*
4. ____________ = *D*
5. ____________ = *E*
6. ______________

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

$$\sqrt[3]{A+B+C+D+E}$$

Solutions to Category 1
Mystery
Meet #2, November, 2002

Answers

1. 53

2. 77¢ or $0.77

3. 18

1. If the product of the two digits is 15, the digits can only be 3 and 5. The number 35 is composite, but **53** is prime.

2. The cashier will not give Misha three $20 bills in change, so the largest bills could be $10 bills. Also, if we use tens, fives, and two dollar bills, it's too much and if we don't use the tens, it's too little. So the bills must be:
$10 + $10 + $10 + $5 + $5 + $5 + $1 + $1 + $1 = $48
That leaves $2.00, or 200¢, for the change in coins and the price of the candy bar. Again, we can eliminate the use of $1 dollar coins and half dollar coins. The four sets of three coins can only be quarters, dimes, nickles, and pennies, with a value of: 25¢ + 25¢ + 25¢ + 10¢ + 10¢ + 10¢ + 5¢ + 5¢ + 5¢ + 1¢ + 1¢ + 1¢ = 123¢ or $1.23. The candy bar must have cost 200¢ – 123¢ = **77¢** or $2.00 – $1.23 = **$0.77**.

3. Since the sum of the values in region B and C is two more than the sum of the values in regions A and C, we know that B must be two more than A. If we took two away from B, then A and B would be the same and the sum of their values would be 26 instead of 28. The value of A must be half of 26 or 13. That means B is 15, two more than 13, and C must be 18, since 33 – 15 = 18. To be sure, we should verify that $A + C = 31$ and indeed 13 + 18 = 31. The largest value is **18**.

Solutions to Category 2
Geometry
Meet #2, November, 2002

Answers

1. 9

2. 24

3. 70

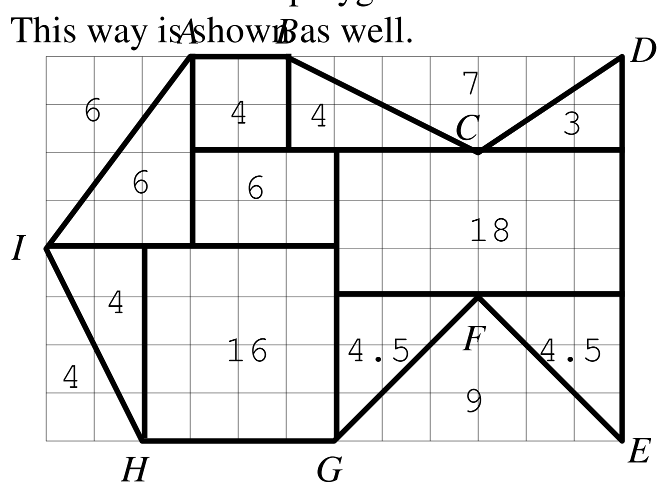
1. The formula for the area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$, where *h* is the height, b_1 is base one, and b_2 is base two. In our case, we know the area and need to find the value of b_1. Substituting the known values into the equation, we have $195.5 = \frac{1}{2} \cdot 17(b_1 + 14)$.
If we had two such trapezoids, then we would have:
$2 \cdot 195.5 = 2 \cdot \frac{1}{2} \cdot 17(b_1 + 14)$ or $391 = 17(b_1 + 14)$.
Dividing both sides of the equation by 17, we get $23 = (b_1 + 14)$, which means the sum of the bases is 23 and base one must be **9** cm.

2. Cutting rectangles from the corners of the sheet of plywood decreases the area but has no affect on the perimeter. The perimeter remains the same as the original 4 ft by 8 ft sheet, which is 4 + 8 + 4 + 8 = **24** feet.

3. The interior of the polygon *ABCDEFGHI* can be subdivided into triangles and rectangles in number of ways. One way is shown below. An alternative solution is to subtract the area of the regions that are outside the polygon from the total of $8 \cdot 12 = 96$ squares in the entire grid. This way is shown as well.

Interior sum:
6 + 4 + 4 + 6 + 16 + 4 + 3 + 18 + 4.5 + 4.5 = 70 square centimeters.

Exterior sum:
6 + 4 + 7 + 9 = 26 and
96 – 26 = 70 square centimeters.



Solutions to Category 3
Number Theory
Meet #2, November, 2002

Answers

1. 1728

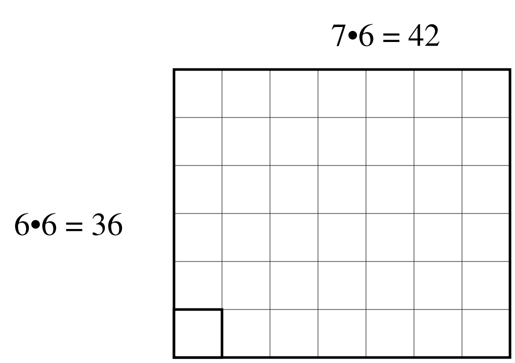
2. 80

3. 6

1. The prime factorization of 36 is $36 = 2^2 \cdot 3^2$ and that of 48 is $48 = 2^4 \cdot 3^1$. Both numbers have two factors of 2 and one factor of 3, so the greatest common factor of 36 and 48 is $2^2 \cdot 3^1 = 12$ and we let F = 12. For the least common multiple of 36 and 48, we need two factors of 3 for the 36 and all four factors of 2 for the 48. The LCM of 36 and 48 is thus $2^4 \cdot 3^2 = 16 \cdot 9 = 144$ and we let M = 144. The product F • M is thus 12 • 144 = **1728**. Notice that the product of 36 and 48 is also 1728. This is true in general: the product of the GCF and the LCM of two numbers is equal to the product of the two numbers.

2. The prime factorization of 135 is $135 = 3^3 \cdot 5^1$ and that of 175 is $175 = 5^2 \cdot 7^1$. To find the largest proper factors of each number, we can simply divide each of the numbers by its smallest prime factor. 135 ÷ 3 = 45 and 175 ÷ 5 = 35. The largest possible value of $n + m$ is thus 45 + 35 = **80**.

3. The largest possible squares that John can cut from his 36 inch by 42 inch poster board have a side length equal to the greatest common factor of 36 and 42, which is **6** inches.



Solutions to Category 4
Arithmetic
Meet #2, November, 2002

Answers

1. 14

2. $\frac{6}{7}$

3. $\frac{11}{13}$

1. Converting the word of to multiplication and all the values to fractions, we get the expression:
$\frac{2}{3} \cdot \frac{40}{100} \cdot \frac{5}{6} \cdot \frac{7}{10} \cdot \frac{90}{1}$. Reducing and cross cancelling, we get:

$$\frac{2}{3} \cdot \frac{2}{5} \cdot \frac{5}{6} \cdot \frac{7}{10} \cdot \frac{90}{1} = \frac{2 \cdot 2 \cdot 5 \cdot 7 \cdot 3 \cdot 3 \cdot 2 \cdot 5}{3 \cdot 5 \cdot 2 \cdot 3 \cdot 2 \cdot 5} = \frac{2 \cdot 7}{1} = \mathbf{14}.$$

2. Repeating decimals can be converted to fractions as follows:

$$\begin{array}{rl} 100x = & 78.\overline{78} \\ - \quad x = & 0.\overline{78} \\ \hline 99x = & 78 \\ x = & \frac{78}{99} \end{array} \qquad \begin{array}{rl} 100y = & 91.\overline{91} \\ - \quad y = & 0.\overline{91} \\ \hline 99y = & 91 \\ y = & \frac{91}{99} \end{array}$$

We can now solve the original problem as follows:

$$\frac{0.\overline{78}}{0.\overline{91}} = \frac{\frac{78}{99}}{\frac{91}{99}} = \frac{78}{99} \cdot \frac{99}{91} = \frac{78}{91} = \frac{6 \cdot 13}{7 \cdot 13} = \mathbf{\frac{6}{7}}$$

3. 65% is the same as the fraction $\frac{65}{100}$, which can be reduced to $\frac{13}{20}$. If the fraction $\frac{20}{39}$ were to increase by 13 parts for every 20 parts, then there would be 13 more 39ths, for a total of $\frac{20+13}{39} = \frac{33}{39}$. This fraction can be reduced by a common factor of 3 to $\mathbf{\frac{11}{13}}$.

Solutions to Category 5
Algebra
Meet #2, November, 2002

Answers

1. 64

2. 3

3. 14.8

1. From the details given, we can write the following equation: $3.5x + 50 = 274$, where *x* is the hourly rate. Subtracting 50 from both sides of the equation, we get: $3.5x = 224$. Dividing both sides of the equation by 3.5, we get $x = 64$. The plumber's hourly rate is **$64**.

2. The verbal sentence translates to the following equation:
$3(x+11) - 5 = 73 - 4(x+6)$. Distributing the multiplication over each addition, we get $3x + 33 - 5 = 73 - 4x - 24$ and then $3x + 28 = 49 - 4x$. Adding 4*x* and subtracting 28 from each side of the equation, this becomes: $7x = 21$. $x = 3$ is the solution to this equation, so the number must be **3**.

3. Substituting 0.5 for *r* and 1.5 for *R* in the formula $S = 4\pi^2 rR$, we get $S = 4 \cdot \pi^2 \cdot 0.5 \cdot 1.5$.
Since $4 \cdot 0.5 \cdot 1.5 = 2 \cdot 1.5 = 3$, the surface of the whole donut is $S = 3 \cdot \pi^2$ and the surface of half the donut would be $\frac{1}{2}S = \frac{3}{2} \cdot \pi^2$. Using 3.14 for π, we get:

$$\frac{3}{2} \cdot 3.14^2 = \frac{3}{2} \cdot 9.8596 = 3 \cdot 4.9298 = 14.7894.$$

Rounding this to the nearest tenth, we can say that about **14.8** square inches of the surface of the donut is covered in chocolate frosting.

Solutions to Category 6
Team Questions
Meet #2, November, 2002

Answers

1. 24

2. 198

3. 108

4. 2

5. 11

6. 7

1. The prime factorization of 792 is $792 = 2^3 \cdot 3^2 \cdot 11^1$. Notice that 792 does not have a prime factor of 5 or 7. We can say this with a zero exponent as follows: $2^3 \cdot 3^2 \cdot 5^0 \cdot 7^0 \cdot 11^1 = 792$. This equation matches the form that we were given $2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e = 792$, so we know that $a = 3$, $b = 2$, $c = 0$, $d = 0$, and $e = 1$. Now we can find the value of $(a+1)(b+1)(c+1)(d+1)(e+1)$, which is $(3+1)(2+1)(0+1)(0+1)(1+1) = 4 \cdot 3 \cdot 1 \cdot 1 \cdot 2 =$ **24**. (Some students may recognize that this calculation gives the number of factors of 792.)

2. It is helpful to notice that 324 is the square of 18. The prime factorizations are $18 = 2^1 \cdot 3^2$ and $324 = 2^2 \cdot 3^4$. Each of the numbers must have at least one factor of 2 and at least two factors of 3 so that the GCF is 18. Moreover, they cannot have more than this one factor of 2 and these two factors of 3 in common or the GCF would be greater than 18. In other words, both numbers must be multiples of 18 and the multipliers must be relatively prime. Also, those multipliers must have a product of 18, so that the LCM will be 324. Let one number be 2 • 18, or 36, and the other be 9 • 18, 0r 162, and all conditions are met. The sum of 36 and 162 is **198**.

3. Whatever the values of *a* and *b*, it is likely that LCM of *a* and *b* is 36, since that appears to be the LCD when the two unit fractions are added. We should find a pair of numbers whose sum is 21 and whose LCM is 36. The numbers 9 and 12 meet those conditions and $\frac{1}{12} + \frac{1}{9} = \frac{3}{36} + \frac{4}{36} = \frac{7}{36}$. The product of 9 and 12 is **108**.

$$
\begin{array}{r}
0.6\overline{428571} \\
14\overline{\smash{)}9.0000000} \\
\underline{-84} \\
60 \\
\underline{-56} \\
40 \\
\underline{-28} \\
120 \\
\underline{-112} \\
80 \\
\underline{-70} \\
100 \\
\underline{-\ \ 98} \\
20 \\
\underline{-14} \\
6
\end{array}
$$

4. We have to divide 9 by 14 using long division and look for a pattern in the repeating decimal. As we see at left, there is a 6 in the tenths place and then the six-digit pattern 428571 begins. The 45th digit in the decimal equivalent of will be the 44th digit in the repeating pattern. Since $6 \cdot 7 = 42$, there will be seven full sets of the pattern and the 44th digit will be the second digit of 428571 or 2.

5. It is obvious that no player can earn 1 or 2 points in the game of bootfall, since the least points you can get is 3. The list below summarizes other scores:

1 not possible
2 not possible
$3 = 1 \cdot 3$
4 not possible
5 not possible
$6 = 2 \cdot 3$
$7 = 1 \cdot 7$
8 not possible
$9 = 3 \cdot 3$
10 not possible
11 not possible
$12 = 4 \cdot 3$
$13 = 2 \cdot 3 + 1 \cdot 7$
$14 = 2 \cdot 7$
$15 = 5 \cdot 3$
$16 = 3 \cdot 3 + 1 \cdot 7$

Once three scores in a row are possible, all scores beyond it are possible by adding more 3's. Thus the highest score that a player can never obtain is 11.

6. Substituting the values for A through E into the expression , we get:

$$\sqrt[3]{A+B+C+D+E} = \sqrt[3]{24+198+108+2+11} = \mathbf{7}$$