Meet # 2
December 2000

Intermediate Mathematics League of Eastern Massachusetts

Meet # 2
December 2000

Version f 12/2008:

Team Q1: specify greater than zero to avoid trivial solution.

Answers moved to page after corresponding problems.

Questions on odd-numbered pages (so if you print double-sided you can get answers on reverse.)

Category 1
Mystery
Meet #2, December 2000

1. John has just purchased five 12-foot planks from which he will cut a total of twenty 34-inch boards for a ramp. If we disregard the thickness of the cut, how many total inches of board will be left over when John is done cutting? (Reminder: 1 foot = 12 inches.)

2. Pick a four-digit number and write it down. Now rearrange the digits to form a second four-digit number. Subtract the smaller number from the larger number and divide the result by 9. What is the remainder?

3. Just for today, the symbol $^{+}(N)^{-}$ signifies the value obtained by alternately adding and subtracting, from least to greatest, each of the positive factors of *N*, not including *N*. For example:
$^{+}(20)^{-} = +1 - 2 + 4 - 5 + 10 = 8$. Find the value of $^{+}(36)^{-}$.

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 1
Mystery
Meet #2, December 2000

Answers

1. 40

2. 0

3. $^{-}11$

1. John will be able to cut four 34-inch boards from each of the five 12-foot planks. Since 12 feet is 144 inches and $4 \times 34 = 136$ inches, John will have five 8-inch boards left over for a total of 40 inches left over.

2. The remainder will be zero regardless of the number used. For example, start with the four-digit number 3285. Rearrange the digits to form 8523. Now subtracting the smaller number from the larger number we obtain: $8523 - 3285 = 5238$. Now dividing by 9, we get: $5238 \div 9 = 582$, remainder 0. In general, any particular digit may move to a different place value and the difference between those place values is itself a multiple of nine. Say the digit d started in the thousands place and moved to the tens place, then we will have the difference $1000d - 10d$, which can be rewritten as $(1000 - 10)d = 990d$. $990d$ is clearly a multiple of 9. A similar argument can be made for each of the four digits used. Another explanation involves the divisibility test for nine. Since the sum of the digits will be the same regardless of their order, we can say that the original number and the rearranged number will have the same remainder when divided by nine. When we subtract one number from the other, that remainder will be lost, leaving a number that is divisible by nine.

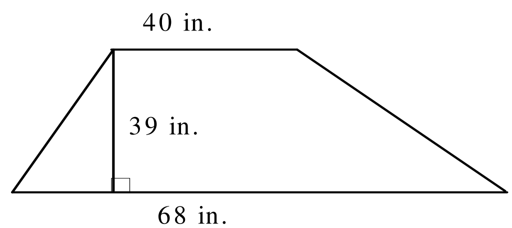
3. Alternately adding and subtracting the proper factors of 36 from least to greatest, we get:

$$^{+}(36)^{-} = +1 - 2 + 3 - 4 + 6 - 9 + 12 - 18 = {}^{-}11$$

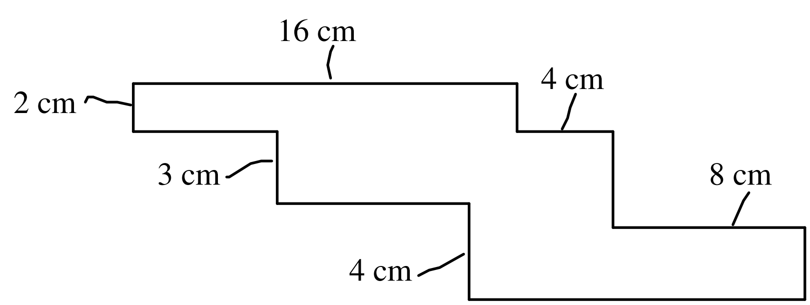
Category 2
Geometry
Meet #2, December 2000

1. The length of the sides of a square and the sides of a regular octagon are the same. The area of the square is 196 square centimeters. What is the perimeter of the octagon? Express your answer in centimeters.

2. The pattern pictured below is to be cut from expensive fabric that comes on a bolt that is 42 inches wide. If Sue buys 2 yards of fabric from the 42-inch bolt and cuts out her pattern, how many square inches of fabric will be wasted? (Reminder: 1 yard = 36 inches.)



3. Find the perimeter of the figure shown below. For this problem, you may assume that all the angles that appear to be right angles are right angles.



Answers
1. ________
2. ________
3. ________

Solutions to Category 2
Geometry
Meet #2, December 2000

Answers

1. 112

2. 918

3. 74

1. A square whose area is 196 square centimeters has a side length of 14 centimeters. Since the octagon has the same side length and all eight of its sides are the same (regular), the perimeter of the octagon is $8 \times 14 = 112$ centimeters.

2. Two yards of fabric from a 42-inch bolt is $72 \times 42 = 3024$ square inches. The area of the trapezoid can be found using the formula $A = \frac{1}{2}h(B_1 + B_2)$, where *h* is the height, B_1 is one of the bases, and B_2 is the other. For Sue's pattern, the area is $A = \frac{1}{2} \times 39 \times (68 + 40) = 2106$ square inches. Since Sue started with 3024 square inches of fabric and only used 2106 square inches, $3024 - 2106 = 918$ square inches of fabric are wasted.

3. Since the figure is "rectilinear" (composed of straight lines and right angles), the total horizontal distance across the top must equal the total horizontal distance across the bottom or $16 + 4 + 8 = 28$ cm. Thus we can say that the three unknown horizontal segments have a sum of 28 cm, even though we don't know their individual lengths. Similarly, the three vertical segments on the right that are not labeled must have the same total length as the three vertical segments on the left that are labeled, namely $2 + 3 + 4 = 9$ cm. The total perimeter of the figure is $28 + 9 + 28 + 9 = 74$ cm.

Category 3
Number Theory
Meet #2, December 2000

1. If $2^l \cdot 3^m \cdot 7^n = 1176$, then find the value of $l^m + m^n + n^l + m^l + n^m + l^n$.

2. If A = the greatest common factor (GCF) of the set of numbers shown below and B = the least common multiple (LCM) of the set, find the difference $B - A$.

$$\{12, 36, 11, 18, 24\}$$

3. The GCF of x and y is 8 and the LCM is 336. If x is 48, then what is the value of y?

Answers

1. _______________
2. _______________
3. _______________

Solutions to Category 3
Number Theory
Meet #2, December 2000

Answers

1. 24

2. 791

3. 56

1. Prime factoring, we find that $1176 = 2^3 \cdot 3^1 \cdot 7^2$, so $l = 3$, $m = 1$, and $n = 2$. Now we carefully substitute these values into the expression $l^m + m^n + n^l + m^l + n^m + l^n$ which becomes $3^1 + 1^2 + 2^3 + 1^3 + 2^1 + 3^2$. Simplifying this, we get $3 + 1 + 8 + 1 + 2 + 9 = 24$.

2. The greatest common factor (GCF) of $\{12, 36, 11, 18, 24\}$ is 1, so $A = 1$. The least common multiple (LCM) of the set is $72 \times 11 = 792$, so $B = 792$. The difference $B - A = 792 - 1 = 791$.

3. The product of the GCF and the LCM of any two numbers is always equal to the product of the two numbers. In this case we have $8 \times 336 = 48y$. Since $336 = 7 \times 48$, we have $8 \times (7 \times 48) = 48y$ or $y = 8 \times 7 = 56$.

Category 4
Arithmetic
Meet #2, December 2000

1. What fraction is 25% greater than $\frac{2}{3}$? Express your answer as a fraction in lowest terms.

2. Find the fraction in lowest terms that is equal to the repeating decimal $0.\overline{24}$.

3. What is the 37^{th} digit in the decimal expansion of $\frac{3}{13}$?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 4
Arithmetic
Meet #2, December 2000

Answers

1. $\frac{5}{6}$

2. $\frac{8}{33}$

3. 2

1. To find the fraction that is 25% greater than 2/3, we could first find out what 25% of 2/3 is and then add that amount to 2/3. Doubling both numerator and denominator of 2/3 gives us 4/6. Now that we have 4 equal parts, we can see that 25% of 4/6 is1/6. Adding 1/6 to 4/6 gives us 5/6.

2. The trick for finding the fraction that equals a repeating decimal is as follows: We set our repeating decimal equal to x and figure that if $x = 0.\overline{24}$ then $100x = 24.\overline{24}$. Next we do a big subtraction which gets rid of the repeating part:

$$\begin{array}{r} 100x = 24.\overline{24} \\ -\quad x = \ 0.\overline{24} \\ \hline 99x = 24 \end{array}$$

Now we can see that $x = \frac{24}{99} = \frac{8}{33}$.

3. All simple fractions will either terminate or repeat and this one will definitely repeat. We don't want to continue dividing $\frac{3}{13}$ out to the 37^{th} decimal place, so we first need to find out what the period is of the repeating decimal. This will have to be done by hand, but students should notice that the pattern begins to repeat after the 6^{th} place. This means we will get the same six digits six times in a row and the 37^{th} digit will be the first digit in a new cycle of the same pattern. The first digit of the pattern is 2.

Category 5
Algebra
Meet #2, December 2000

1. Nine times the sum of a number and fourteen is three more than ten times the difference when the number is subtracted from fifty-six. Find the number.

2. The formula for the surface area of a cone including the base is $A = \pi rs + \pi r^2$, where r is the radius of the base and s is the slant height of the cone. Find the surface area of a cone whose base has a radius of 8 cm and whose slant height is 10 cm. Leave your answer in terms of π.

3. If you add $\frac{1}{4}$ of the price of a candy bar to $\frac{1}{2}$ of the change you would get if you bought the candy bar with a dollar (no tax), you get the exact price of the candy bar. How much is the candy bar?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 5
Algebra
Meet #2, December 2000

Answers

1. 23

2. 144π

3. \$0.40 or 40¢

1. Translating the English into algebra, we get:

$$9(x+14) = 3+10(56-x)$$
$$9x+126 = 3+560-10x$$
$$19x = 437$$
$$x = 23$$

2. Substituting $r = 8$ and $s = 10$ into the formula for the surface area of a cone (with base), we get:

$$A = \pi \times 8 \times 10 + \pi \times 8^2 = 80\pi + 64\pi = 144\pi$$

3. Once again our first step is to translate the English to algebra, then we just solve for *x*.

$$\frac{1}{4}x + \frac{1}{2}(1.00 - x) = x$$
$$\frac{1}{4}x + 0.50 - \frac{1}{2}x = x$$
$$0.50 - \frac{1}{4}x = x$$
$$0.50 = \frac{5}{4}x$$
$$x = \frac{4}{5} \times 0.50 = 0.40$$

The candy bar costs \$0.40 or 40¢.

Category 6
Team Questions
Meet #2, December 2000

1. Tom and Brenda each have a positive whole number of dollars. Four fifths of Tom's money is equal to five sevenths of Brenda's money. If Tom and Brenda each have less than \$50, how much do they have together?

2. Let's agree that a number with three of the same digits in a row will be called a triplet. Let's also agree that a number with four of the same digits in a row will be called a quadruplet and cannot be counted as a triplet. How many triplets are there between 1000 and 2000 (excluding 1000 and 2000)?

3. Express the complex fraction shown to the right as a simplified mixed number.

$$9+\cfrac{8}{7+\cfrac{6}{5+\cfrac{4}{3+\cfrac{2}{1}}}}$$

4. The "proper" factors of a number are all the factors except the number itself. Find the sum of the numbers between 20 and 30 for which the *product* of the proper factors of the number equals the number itself.

5. Find x and y so that $4+x=4x$ and $5+y=5y$. What is the value of $x-y$?

Answers

1. __________ = A
2. __________ = B
3. __________ = C
4. __________ = D
5. __________ = E
6. ____________

6. Evaluate the following expression, using the answers to questions 1 through 5 as the values of *A, B, C, D,* and *E*, respectively:

$$DE+\left(\sqrt[3]{C+\frac{12E}{4(A+1)+B}+B}\right)$$

Solutions to Category 6
Team Questions
Meet #2, December 2000

Answers

1. \$53

2. 17

3. $9\frac{232}{233}$

4. 96

5. $\frac{1}{12}$

6. 11

1. The second sentence tells us that $\frac{4}{5}T = \frac{5}{7}B$, where *T* is the amount that Tom has and *B* is the amount that Brenda has. This is one equation with two unknowns, so there are infinitely many combinations of *T* and *B* that will work. We do know, however, that they each have a whole number of dollars under \$50. The trick here is to look at equivalent fractions for $\frac{4}{5}$ and $\frac{5}{7}$ until we find a lowest common *numerator*. The LCM of 4 and 5 is 20, so we rewrite our first equation with this common numerator: $\frac{20}{25}T = \frac{20}{28}B$. Now it's easier to see that Tom has \$25 when Brenda has \$28. These are least whole numbers that will work. *T* = \$50 and *B* = \$56 would also work, but they aren't under \$50. Together Tom and Brenda must have \$25 + \$28 = \$53.
** The original problem allowed Tom and Brenda to have ZERO dollars, which would be a very quick trivial solution. The problem was revised for this archive to what was intended.*

2. Since 1000 is not to be included in our count of triplet numbers, the least triplet we encounter is 1110. We cannot include 1111 in our count since that is a quadruplet number. 1112 through 1119 give us eight more triplets. Working our way up, we encounter 1222, then 1333, then 1444, etc. until 1999. That gives another eight triplets and it's the end of our count, since 2000 is not to be included either. Our total is $1 + 8 + 8 = 17$.

3. To simplify this sort of fraction, one must start at the bottom and work back up.

$$9+\cfrac{8}{7+\cfrac{6}{5+\cfrac{4}{3+\cfrac{2}{1}}}}=9+\cfrac{8}{7+\cfrac{6}{5+\cfrac{4}{5}}}=9+\cfrac{8}{7+\cfrac{6}{\frac{29}{5}}}$$

$$=9+\cfrac{8}{\frac{233}{29}}=9+\frac{232}{233}=9\frac{232}{233}.$$

4. Some experimentation will reveal that prime numbers yield a product of 1 and numbers with lots of factors yield a product greater than the number itself. The first number between 20 and 30 for which the product of its proper factors equals itself is 21 ($1\times 3\times 7=21$). 22 also works ($1\times 2\times 11=22$). 23 does not work because its only proper factor is 1. 24 has too many factors so it yields a huge product. In the end, we find that only 21, 22, 26, and 27 work. Their sum is 96.

5. If $4+x=4x$, then $4=3x$ and $x=\frac{4}{3}$. Similarly, if $5+y=5y$, then $5=4y$ and $y=\frac{5}{4}$.

We have to find $x-y$, so $\frac{4}{3}-\frac{5}{4}=\frac{16}{12}-\frac{15}{12}=\frac{1}{12}$.

6. Substituting for *A* through *E* gives:

$$96\times\frac{1}{12}+\left(\sqrt[3]{9\frac{232}{233}+\frac{12\times\frac{1}{12}}{4(53+1)+17}+17}\right)$$

$$=8+\left(\sqrt[3]{9\frac{232}{233}+\frac{1}{216+17}+17}\right)=8+\sqrt[3]{27}=11$$